A NOTE ON THE FIBONACCI GROUPS

BY

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ABSTRACT

The so-called Fibonacci groups were introduced by Conway in [1]; a more general class of groups is investigated in [2]. In this article, we consider the derived factor-groups of the groups in the latter class, and use two different presentations of them, first to prove that they are all finite and second to give formulae for their orders.

1. Introduction

The so-called Fibonacci groups were first introduced by Conway in 1965 (see [1]) and their properties are discussed at length in [2] which also contains a fairly complete bibliography. Apart from motivation and a few elementary results of linear algebra, the present article is self-contained.

Let F be the free group on the set $\{x_1, \cdots, x_n\}$, γ the permutation $(12 \cdots n)$ and $\bar{\gamma}$ the automorphism of F given by

$$x_i\bar{\gamma} = x_{i\gamma}, \qquad 1 \leqq i \leqq n.$$

Letting w_r be the word

$$x_1 \cdots x_r \, x_{r+1}^{-1}$$

in F (all subscripts being reduced modulo n to lie in the set $\{1, \cdots, n\}$), we set R equal to the normal closure of the elements

$$w_r\bar{\gamma}^{\,i}, \qquad 0 \leqq i \leqq n-1,$$

and define

$$F(r,n) = F/R,$$

the Fibonacci group of type (r,n). We denote the derived factor group of $F(r,n)$ by $A(r,n)$.

The purpose of this note is to derive two relation matrices for $A(r,n)$, the first

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with respect to the generators $x_1, \cdots, x_r$ and the second with respect to the generators $x_1, \cdots, x_n$. From the first of these we prove that $A(r,n)$ is finite whenever $r > 1$ ($F(1,n)$ is clearly infinite cyclic for all n), and from the second we derive a formula for its order. To avoid triviality, we assume $r > 1$ throughout.

2. The presentation on r generators

For a fixed natural number r, we define a sequence $\{\boldsymbol{a}_i\}$ of r-tuples over $\boldsymbol{Z}$ (the integers) as follows:

i. for $1 \leqq i \leqq r$,

$$\boldsymbol{a}_i = (\delta_{i1}, \cdots, \delta_{ir}),$$

where δ_{ij} is the Kronecker delta, and

ii. for $i > r$,

$$\boldsymbol{a}_i = \sum_{j=i-r}^{i-1} \boldsymbol{a}_j,$$

where addition is componentwise.

Under the isomorphism

$$\sum_{j=1}^{r} \alpha_j x_j \leftrightarrow (\alpha_1, \cdots, \alpha_r), \ \alpha_i \in \boldsymbol{Z}, \qquad 1 \leqq i \leqq r,$$

between the free (additively-written) abelian group on $x_1, \cdots, x_r$ and the group of r-tuples over $\boldsymbol{Z}$, the relations defining $A(r,n)$ pass to the relations

$$\boldsymbol{a}_{n+j} = \boldsymbol{a}_j, \qquad 1 \leqq j \leqq r.$$

Thus, letting $M_n (n \geqq 0)$ be the $r \times r$ matrix whose jth row is the vector $\boldsymbol{a}_{n+j}$ $(1 \leqq j \leqq r)$, a relation matrix for $A(r,n)$ is

$$M_n - I,$$

where I is the r-rowed identity matrix. But since, for all $n \geqq 0$,

$$M_1 \begin{bmatrix} \boldsymbol{a}_{n+1} \\ \vdots \\ \boldsymbol{a}_{n+r} \end{bmatrix} = \begin{bmatrix} \boldsymbol{a}_{n+2} \\ \vdots \\ \boldsymbol{a}_{n+r+1} \end{bmatrix},$$

we see at once that $M_n = M_1^n$ for all $n \geqq 0$. Letting $M = M_1$, we have the following theorem (which is [2, Th. 7] and generalises Bumby's result [1] for $r = 2$).

THEOREM 1. *A relation matrix for* $A(r, n)$ *is*

$$M^n - I,$$

where M *is the companion matrix of the polynomial*

$$p_r(\lambda) = \lambda^r - \sum_{i=0}^{r-1} \lambda^i,$$

namely

$$M = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ & & & \cdots & & \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{bmatrix} .$$

THEOREM 2. *For* $r > 1$, *the polynomial* $p_r(\lambda)$ *has no root on the unit circle.*

PROOF. For a contradiction, suppose that ξ is a complex number of unit modulus such that $p_r(\xi) = 0$. Multiplying by $(1 - \xi)$,

$$(\xi^r - \xi^{r+1}) - (1 - \xi^r) = 0,$$

whence

$$2\xi^r = 1 + \xi^{r+1}, \tag{1}$$

so that, by hypothesis,

$$|1 + \xi^{r+1}| = 2 = |1| + |\xi^{r+1}|.$$

This means that ξ^{r+1} is real and non-negative, so that $\xi^{r+1} = 1$ since $|\xi| = 1$. From (1) we see that $\xi^r = 1$, whence $\xi = 1$ and

$$0 = P_r(\xi) = r - 1,$$

which is the required contradiction.

COROLLARY 1. *For* $r \neq s$, *the polynomials* $p_r(\lambda)$ *and* $p_s(\lambda)$ *have no root in common.*

PROOF. This follows at once from the theorem, together with the identity

$$\lambda^{s-r} p_r(\lambda) - p_s(\lambda) = \sum_{i=0}^{s-r-1} \lambda^i,$$

valid for all $s > r$.

COROLLARY 2. *$A(r, n)$ is a finite group.*

PROOF. The characteristic polynomial of M is just $p_r(\lambda)$, and so no eigenvalue of M is a root of unity by Theorem 2. But the eigenvalues of M^n are just the nth powers of the eigenvalues of M, so that 1 is never an eigenvalue of M^n, that is,

$$\det(M^n - I) \neq 0,$$

for all r, n. By Theorem 1, this is precisely the condition that $A(r,n)$ be finite.

Letting δ_i be the highest common factor of the i-rowed minors of $M^n - I$, $1 \leqq i \leqq r$, $\delta_0 = 1$, the invariant factors of $A(r,n)$ for $r > 1$ are

$$\{|\delta_i/\delta_{i-1}| \mid 1 \leqq i \leqq r\} \tag{2}$$

and in particular

$$|A(r,n)| = |\det(M^n - I)|.$$

Letting $\xi_1, \cdots, \xi_r$ be the roots of $p_r(\lambda)$, we thus have

$$|A(r,n)| = \left|\prod_{i=1}^{r} (\xi_i^n - 1)\right|. \tag{3}$$

Using (2), the invariant factors of $A(r,n)$ are tabulated in [2], for $r = 3, 4$, $n \leqq 20$. A more tractable formula for $|A(r,n)|$ is given in the next section.

3. The presentation on n generators

As in Section 2, we let the elements of the free abelian group on the set $\{x_1, \cdots, x_n\}$ correspond to n-tuples over $\boldsymbol{Z}$ in the obvious way. The word w_r then corresponds to the n-tuple $(c_1, \cdots, c_n)$, where

$$c_i = \begin{cases} k+1, & i \leqq s, \\ k-1, & i = s+1, \\ k, & i > s+1, \end{cases} \tag{4}$$

where

$$r = kn + s, \qquad 0 \leqq s < n. \tag{5}$$

The images of the other relators are clearly obtained by cyclic permutation of the components of $(c_1, \cdots, c_n)$.

THEOREM 3. *A relation matrix for $A(r,n)$ is the circulant matrix C whose first row is $(c_1, \cdots, c_n)$ where the c_i are given by* (4).

The determinant of this circulant matrix is well known to be equal to

$$\pm \prod_{i=1}^{n} \left(\sum_{j=1}^{n} c_j \omega_i^{j-1}\right),$$

where $\omega_1 = 1, \omega_2, \cdots, \omega_n$ are the distinct nth roots of unity. Hence, in our case,

$$\det C = \pm \prod_{i=1}^{n} ((1 + \omega_i + \cdots + \omega_i^{s-1} - \omega^s) + k(1 + \omega_i + \cdots + \omega_i^{n-1}))$$

$$= \pm (r-1) \prod_{i=2}^{n} (1 + \omega_i + \cdots + \omega_i^{s-1} - \omega_i^s).$$

Thus, since $A(r, n)$ is finite (Corollary 2 above),

$$|A(r,n)| = \left|(r-1) \prod_{i=2}^{n} (1 + \omega_i + \cdots + \omega_i^{s-1} - \omega_i^s)\right|. \tag{6}$$

Fixing the notation k, s as in (5), we have the following two corollaries.

COROLLARY 3. i. *If* $s \neq 1$,

$$|A(r,n)| = \frac{r-1}{s-1}|A(s,n)|.$$

ii. *If* $s = 1$,

$$|A(r,n)| = n(r-1).$$

PROOF. i. This follows at once from (6).

ii. Letting

$$q_n(\lambda) = 1 + \lambda + \cdots + \lambda^{n-1} = \prod_{i=2}^{n} (\lambda - \omega_i)$$

we know that

$$q_n(\omega_i) = \begin{cases} n & \text{if } i = 1, \\ 0 & \text{if } 2 \leqq i \leqq n. \end{cases} \tag{7}$$

When $s = 1$, formula (6) reduces to

$$|A(r,n)| = (r-1)|q_n(1)| = n(r-1),$$

as required.

COROLLARY 4. i. *If* $s = 0$, $|A(r,n)| = r - 1$.

ii. *If* $s = n - 1$, $|A(r,n)| = (r-1)2^{n-1}$.

iii. *If* $s = n - 2$, $|A(r,n)| = \frac{1}{3}(r-1)(2^n + (-1)^{n+1})$.

PROOF. i. This follows at once from (6). The other parts follow from (6) and (7).

ii. $$|A(r,n)| = (r-1)\left|\prod_{i=2}^{n}(1+\omega_i\cdots+\omega_i^{n-2}-\omega_i^{n-1})\right|, \text{ by (6),}$$

$$= (r-1)\left|\prod_{i=2}^{n}(-2\omega_i^{n-1})\right|, \text{ using (7),}$$

$$= (r-1)2^{n-1}.$$

iii. $$|A(r,n)| = (r-1)\left|\prod_{i=2}^{n}(1+\omega_i+\cdots+\omega_i^{n-3}-\omega_i^{n-2}\right|, \text{ by (6),}$$

$$= (r-1)\left|\prod_{i=2}^{n}(-2\omega_i^{n-2}-\omega_i^{n-1})\right|, \text{ by (7),}$$

$$= (r-1)\cdot 1\cdot\left|\prod_{i=2}^{n}(-2-\omega_i)\right|$$

$$= (r-1)\,|q_n(-2)|$$

$$= (r-1)\left|\frac{1-(-2)^n}{1-(-2)}\right|$$

$$= \tfrac{1}{3}(r-1)\,(2^n+(-1)^{n+1}),$$

which, as luck would have it, is always a positive integer.

References

1. J. H. Conway, *Solution to advanced problem* 5327, Amer. Math. Monthly **74** (1967), 91–93.

2. D. L. Johnson, J. W. Wamsley and D. Wright, *The Fibonacci groups*, to appear in Proc. London Math. Soc.

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